REFERENCE

1. Strutt, J. W. (Rayleigh 3rd Baron), The Theory of Sound, 2nd ed. London, Macmillan, 1894.

Translated by L. K.

UDC 531.31

ON THE EXISTENCE CONDITIONS FOR THE PARTICULAR JACOBI INTEGRAL

PMM Vol. 40, № 4, 1976, pp. 611-617
I. A. KEIS
(Tallin)
(Received July 4, 1975)

For certain nonholonomic and holonomic mechanical systems we have obtained the existence conditions for the particular integral being a linear bundle of the Hamiltonian and the momenta. The conditions are simplified for a certain class of holonomic systems, containing the well-known [1] case of the particular Jacobi integral. An example of the fulfillment of these conditions is a variant of the restricted problem of translation-rotational motion of a gyrostat in a Newtonian force field.

1. We consider a mechanical system S with the Lagrangian $L = T + U + N$. The linear function N of generalized velocities is the Meyer potential [2—4] of certain electromagnetic and gyroscopic forces. We separate the system S with the position coordinate vector $\mathbf{y} = (x_i, z_r)^*$ into subsystems S' and S'' with vectors $\mathbf{x} = (x_i)^*$ and $\mathbf{z} = (z_r)^*$

$$i = 1, 2, \ldots, l;\ r = 1, 2, \ldots, p;\ 1 \leqslant l,\ 1 \leqslant p;\ \dim y = l + p = n$$

We write the Lagrangian of system S as the sum

$$L = L_2' + L_1' + L_2'' + L_1'' + L^* + L_0 \tag{1.1}$$
$$L_2' = {}^1/_2 l_{ij}'(t, \mathbf{y})\, x_i^{\boldsymbol\cdot} x_j^{\boldsymbol\cdot},\quad l_{ij}' = l_{ji}',\quad \|l_{ij}'\| > 0\quad (i, j = 1, 2, \ldots, l)$$
$$L_1' = l_j'(t, \mathbf{y})\, x_j^{\boldsymbol\cdot},\quad L_2'' = {}^1/_2 l_{rs}''(t, \mathbf{y})\, z_r^{\boldsymbol\cdot} z_s^{\boldsymbol\cdot},\quad l_{rs}'' = l_{sr}''\ (r, s = 1, 2, \ldots, p$$
$$L_1'' = l_r(t, \mathbf{y})\, z_r^{\boldsymbol\cdot},\quad L^* = l_{ir}(t, \mathbf{y})\, x_i^{\boldsymbol\cdot} z_r^{\boldsymbol\cdot},\quad L_0 = L_0(t, \mathbf{y}),\quad f^{\boldsymbol\cdot} = df/dt$$

Here and below summation is carried out over like indices and the superscript zero signifies the result of a substitution

$$f^\circ = f^\circ(t, \mathbf{x}, \mathbf{x}^{\boldsymbol\cdot}) = f(t, \mathbf{x}, \mathbf{x}^{\boldsymbol\cdot}, \mathbf{r}(t), \mathbf{v}(t))$$
$$(\mathbf{z} = \mathbf{r}(t),\ \mathbf{z}^{\boldsymbol\cdot} = d\mathbf{r}/dt = \mathbf{v}(t))$$

We denote $\mathbf{r}(t)$, $\mathbf{v}(t)$ as the known motion of subsystem S'', for which the cylinder $\mathbf{z} = \mathbf{r}(t)$, $\mathbf{z}^{\boldsymbol\cdot} = \mathbf{v}(t)$ is an invariant set of motions of S. In particular, the motion $\mathbf{r}_*$, $\mathbf{r}_*^{\boldsymbol\cdot}$ of subsystem S'' possesses this property if system S has the particular invariants h_s $(s, r = 1, 2, \ldots p)$

$$h_s(t, \mathbf{z}) = 0,\quad \det \| \partial h_s / \partial z_r \| \neq 0\quad (h_s(t, \mathbf{r}_*(t)) \equiv 0)$$

or if the motions of S' have no effect on S''.

We restrict consideration to the system satisfying the conditions

$$(\partial l_{is}{}^* / \partial x_j)^\circ = 0, \quad (l_{is}{}^* v_s)^\circ = 0 \quad (i, j = 1, 2, \ldots, l)$$
$$(\partial l_{ir}{}^* / \partial z_s + \partial l_{is}{}^* / \partial z_r)^\bullet = 0, \quad (\partial L / \partial t)^\circ = 0 \quad (r, s = 1, 2, \ldots, p) \tag{1.2}$$

along the motion $\mathbf{r}\ (t)$, $\mathbf{v}\ (t)$ of subsystem S''. The last group of equalities in (1.2) we shall call the autonomy conditions. They are satisfied, for instance, when $\partial L / \partial t \equiv 0$. The remaining conditions are satisfied, in particular, when $L^* \equiv 0$.

Let us assume that S is subject only to the ideal linear nonholonomic constraints

$$a_{\alpha i}(t, \mathbf{y})\, \delta x_i + b_{\alpha s}(t, \mathbf{y})\, \delta z_s = 0 \quad (\alpha = 1, 2, \ldots, m \leqslant n - 1)$$

Then along $r\ (t)$, $v\ (t)$ we have the conditions on the virtual displacements

$$a_{\alpha i}{}^\circ \delta x_i + b_{\alpha s}{}^\circ \delta z_s' = 0 \quad (a_{\alpha i}{}^\circ = a_{\alpha i}(t, \mathbf{x}, \mathbf{r}),\ b_{\alpha s}{}^\circ = b_{\alpha s}(t, \mathbf{x}, \mathbf{r})) \tag{1.3}$$

If the system is holonomic, $m = 0$. We assume that the variations

$$\delta y = \varepsilon\, (x_i^\bullet + u_i(t), v_s(t))^* \quad \Big(\sum_{i=1}^{l} u_i^2 \neq 0\Big) \tag{1.4}$$

satisfy Eqs. (1.3) for arbitrary sufficiently smooth functions $u_i\ (t)$. Then the variations (1.4) are the virtual displacements of system S along the motion $r\ (t)$, $v\ (t)$ of subsystem S''. This condition is satisfied when

$$a_{\alpha i}{}^\circ \equiv 0,\ b_{\alpha s}{}^\circ v_s \equiv 0,\ \operatorname{rank} \| b_{\alpha s}{}^\circ \| = m \leqslant p - 1$$

We assume that the nonpotential generalized forces $Q_i\ (t, \mathbf{y}, \mathbf{y}^\bullet)$ and $Q_s'(t, \mathbf{y}, \mathbf{y}^\bullet)$ do not act on displacements (1.4), i.e.

$$(x_i^\bullet + u_i)\, Q_i^\circ + v_s Q_s'^\circ = 0 \tag{1.5}$$

We examine only the mechanical systems S for which displacements (1.4) are virtual and conditions (1.2) and (1.5) are satisfied. We call them Jacobi mechanical systems. A holonomic system [1] satisfies conditions (1.2) and (1.5).

2. Let us consider the expression, linear with respect to the Hamiltionian and the momenta, of the Jacobi invariant type [1]

$$I = (L_2' + L_2'' - L_0)^\circ + u_i(t)\, (\partial L / \partial x_i^\bullet)^\circ - \int_0^t h(\tau)\, d\tau \tag{2.1}$$

where $h\ (\tau)$ is an arbitrary sufficiently smooth function. Let us determine the conditions which the Jacobi mechanical system and the functions u_i and h must satisfy in order for I to be an invariant of the motion of S along the trajectory $\mathbf{z} = \mathbf{r}\ (t)$, $\mathbf{z}^\bullet = \mathbf{v}\ (t)$ of subsystem S''. From the general equation of dynamics

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial x_i^\bullet}\right) - \frac{\partial L}{\partial x_i} - Q_i\right]\delta x_i + \left[\frac{d}{dt}\left(\frac{\partial L}{\partial z_s^\bullet}\right) - \frac{\partial L}{\partial z_s} - Q_s\right]\delta z_s = 0$$

with due regard to (1.1) — (1.5) and (2.1), we obtain the equality

$$dI / dt = d_{ik}(t, x)\, x_i^\bullet x_k^\bullet + d_k(t, x)\, x_k^\bullet + d_0(t, x) \tag{2.2}$$
$$d_{ik} = u_j \partial l_{ik} / \partial x_j, \quad l_{ik} = (l_{ik}')^\circ = l_{ki}, \quad \| l_{ik} \| > 0$$
$$d_k = u_j \partial l_k / \partial x_j + u_i^\bullet l_{ik}, \quad l_k = (l_k')^\circ \quad (i, j, k = 1, 2, \ldots, l)$$

$$d_0 = u_j \partial R^\circ / \partial x_j - \partial R^\circ / \partial t + u_i^\cdot l_i + v_s^\cdot (\partial K / \partial z_s)^\circ - h(t)$$
$$R = L_2'' + L_1'' + L_0, \quad R^\circ = (L_2'' + L_1'' + L_0)^\circ, \quad K = L_2'' + L_1''$$
$$(s, r = 1, 2, \ldots, p)$$

For the particular invariant (2.1) of the motion of system S to exist it is necessary and sufficient to satisfy the equations

$$X_1(l_{ik}) = 0 \quad (X_1(f) = u_j \partial f / \partial x_j, \quad i, j, k = 1, 2, \ldots, l) \qquad (2.3)$$
$$X_1(l_k) = -u_i^\cdot l_{ik}$$
$$X_2(R^\circ) = h - u_i^\cdot l_i - v_s^\cdot (\partial K / \partial z_s)^\circ \quad (X_2 = X_1 - \partial / \partial t)$$

for which the quantity (2.2) vanishes. With due regard to (1.2) and to the relation

$$\partial f^\circ / \partial t = (\partial f / \partial t)^\circ + v_s (\partial f / \partial z_s)^\circ + v_s^\cdot (\partial f / \partial z_s^\cdot)^\circ$$

we write the autonomy conditions in (1.2) in the following form:

$$\partial l_{ik} / \partial t = v_s (\partial l_{ik}' / \partial z_s)^\circ \quad (i, k = 1, 2, \ldots, l) \qquad (2.4)$$
$$\partial l_k / \partial t = v_s (\partial l_k' / \partial z_s)^\circ + v_s^\cdot (l_{ks}^*)^\circ$$
$$\partial R^\circ / \partial t = v_s (\partial R / \partial z_s)^\circ + v_s^\cdot (\partial R / \partial z_s^\cdot)^\circ$$

From the first groups of equations in (2.3) and in (2.4) we have the system

$$X_1(f_{ik}) = 0, \quad X_{ik}(f_{ik}) = 0 \quad (i, k = 1, 2, \ldots, l) \qquad (2.5)$$
$$X_{ik} = \partial / \partial t + w_{ik}(t, \mathbf{x}) \partial / \partial f, \quad w_{ik} = v_s (\partial l_{ik}' / \partial z_s)^\circ, \quad f = l_{ik}$$

where the $l_{ik}(t, x)$ satisfy the equalities $f_{ik}(t, \mathbf{x}, l_{ik}) = 0$. For simplicity we assume that Eqs. (2.5) comprise a complete system [5, 6]. For this it is necessary and sufficient that each commutator $Z_{ik} = X_{ik}(X_1) - X_1(X_{ik})$ satisfies the equality $Z_{ik} = \lambda X_1 + \mu X_{ik}$ ($\lambda = \lambda_{ik}(t \ \mathbf{x} \ f)$, $\mu = \mu_{ik}(t, \mathbf{x}, f)$) with arbitrary functions λ and μ. Hence we obtain the equations

$$u_i^\cdot = \lambda(t) u_i, \quad v_s \left(\frac{\partial}{\partial z_s} X_1(l_{ik}') \right)^\circ = 0 \qquad (2.6)$$
$$\lambda_{ik}(t, \mathbf{x}, f) \equiv \lambda(t), \quad \mu_{ik}(t, \mathbf{x}, f) \equiv 0 \quad (i, k = 1, 2, \ldots, l; \ s = 1, 2, \ldots p)$$

Substituting the general solution of the first group of equations in (2.6)

$$u_i = c_i w(t), \quad c_i = \text{const}, \quad w = \exp\left[\int_0^t \lambda(\tau) d\tau\right] \quad (c = (c_i)^* \neq 0) \qquad (2.7)$$

into the second group, we obtain the compatability conditions for the first groups of equations in (2.3) and (2.4)

$$v_s' \left(\frac{\partial}{\partial z_s} X(l_{ik}') \right)^\circ = 0 \quad \left(X = c_j \frac{\partial}{\partial x_j}; \ i, j, k = 1, 2, \ldots, l \right) \qquad (2.8)$$

In order to satisfy the compatability conditions (2.8) and the first group of equations in (2.3) it is sufficient to assume $\xi_1 = c_k x_k$ as an ignorable coordinate of function L_2'

$$X(l_{ik}') = 0 \quad (i, j, k = 1, 2, \ldots, l) \qquad (2.9)$$

Let us accept conditions (2. 9). Then for l_{ik}' we have the expressions

$$l_{ik}' = m_{ik}(t, \xi', \mathbf{z}) \quad (m_{ik} = m_{ki}, \| m_{ik} \| > 0) \qquad (2.10)$$

in which the coordinates of vector $\xi' = (\xi_2, \ldots, \xi_l)^*$ are specified by a transformation $\xi' = Px$ of the form

$$\xi_m = p_{mi}x_i, \quad p_{mi} = \text{const}, \quad P = \| p_{mi} \| \quad (m = 2, 3, \ldots, l) \qquad (2.11)$$
$$p_{mi}c_i = 0, \qquad \text{rank}\, P = l - 1$$

By virtue of (2. 11) the functions $m_{ik}(t, P\mathbf{x}, \mathbf{z})$ satisfy the first group of equations in (2.4).

With due regard to (2. 7) and (2. 10) the remaining equations in (2. 3) take the form

$$X(l_k) + \lambda(t)c_j m_{kj}^\circ = 0 \quad (m_{kj}^\circ = m_{kj}(t, \xi', \mathbf{r}(t)) = m_{jk}^\circ) \qquad (2.12)$$
$$X(R^\circ) + \lambda(t)\, c_j l_j - v_s (\partial R / \partial z_s)^\circ - H = 0 \quad (H(t) = hw^{-1})$$
$$k, j = 1, 2, \ldots, l; \; s = 1, 2, \ldots, p$$

Analogously to the preceding we obtain the compatability conditions for Eqs. (2. 12) and (2.4) which are expressed by the equalities

$$v_s X (\partial l'_k / \partial z_s^{\cdot})^\circ + c_j \partial (\lambda m_{kj}^\circ) / \partial t = 0 \qquad (2.13)$$
$$v_s X^1 (\partial R / \partial z_s^{\cdot})^\circ + c_j \partial (\lambda l_j) / \partial t + v_s^{\cdot} R_s^1 - H^{\cdot} = 0.$$
$$X^1 = c_j \partial / \partial x_j - \partial / \partial t, \quad R_s^1 = X (\partial R / \partial z_s^{\cdot})^\circ - (\partial R / \partial z_s)^\circ$$
$$k, j = 1, 2, \ldots, l; \quad s = 1, 2, \ldots, p$$

Formulas (2. 2), (2. 3), (2. 7), (2. 10) and (2. 11) convey the sense of the notation adopted. By combining the assumptions made, we obtain the following statement.

If functions $\lambda^\circ(t)$ and $h^\circ(t)$ and constants c_k° ($\mathbf{c}^\circ \neq 0$) exist for which the equalities (2. 9), (2. 12) and (2. 13) are satisfied for the Jacobi mechanical system, then system S has the invariant

$$I = (L_2' + L_2'' - L_0)^\circ + c_k^\circ w^\circ(t) (\partial L / \partial x_k^{\cdot})^\circ - \int_0^t h^\circ(\tau)\, d\tau \qquad (2.14)$$

along the motion $\mathbf{r}(t) = \mathbf{z}$, $\mathbf{v}(t) = \mathbf{z}^{\cdot}$. Expression (2. 1) takes the form (2. 14) on the strength of equalities (2. 6). We note that equalities (2. 4), equivalent to the autonomy conditions in (1. 2), are satisfied by the Jacobi mechanical system by definition. The statement is preserved if assumptions (1. 4) and (1. 5) are replaced by the following. It is sufficient that the variations $\delta^\circ y = \varepsilon (x_k^{\cdot} + c_k^\circ w^\circ(t), v_s(t))^*$ satisfy Eqs. (1. 3) and that the equality

$$(x_k^{\cdot} + c_k^\circ w^\circ(t))\, Q_k^\circ + v_s(t)\, Q_s'^\circ = 0 \quad \left(w^\circ(t) = \exp \int_0^t \lambda^\circ(\tau)\, d\tau\right)$$

be satisfied for the nonpotential generalized forces.

3. The existence conditions for the particular invariant (2. 14) simplify for a holonomic potential system S^* satisfying the following conditions. Let the functions (1. 1) for system S^* satisfy the identities

$$L^* \equiv 0, \quad \partial L / \partial t \equiv 0, \quad \partial L_\alpha' / \partial z_s \equiv 0, \quad \partial L_\alpha'' / \partial x_k \equiv 0 \qquad (3.1)$$
$$\alpha = 1, 2; \; s = 1, 2, \ldots, p; \; k = 1, 2, \ldots, l$$

We assume that to the motion $z = r(t)$, $z^{\cdot} = v(t)$ of subsystem S_2^* there corresponds an invariant set of motions of S^*, namely, the cylinder $z = r(t)$, $z^{\cdot} = v(t)$. Since conditions (1.2), (1.4) and (1.5) are satisfied, system S^* is a Jacobi mechanical system.

Let the coordinate $\xi_1 = c_k x_k$ be ignorable for L_2' and L_1'

$$c_k \partial L_2' / \partial x_k \equiv 0, \quad c_k \partial L_1' / \partial x_k \equiv 0 \quad (c_k = \text{const}, \quad c \neq 0) \tag{3.2}$$

We set $\lambda(t) \equiv 0$. By virtue of identities (3.1) and (3.2) conditions (2.12) and (2.13) are reduced to the two equalities

$$X(L_0)^\circ - v_s\left(\frac{\partial R}{\partial z_s}\right)^\circ - h = 0 \tag{3.3}$$

$$\left(X = c_j \frac{\partial}{\partial x_j}, \; R = L_2'' + L_1'' + L_0\right)$$

$$X^1\left(v_s \frac{\partial L_0}{\partial z_s}\right)^\circ - \frac{d}{dt}\left(v_s \frac{\partial K}{\partial z_s}\right)^\circ - h^{\cdot} = 0 \quad \left(X^1 = X - \frac{\partial}{\partial t}, \; K = L_2'' + L_1''\right)$$

Using the statement obtained, we arrive at the following conclusion for system S^*. If equalities (3.2) and (3.3) are satisfied for $c_k = c_k^*$ and $h(t) = h^*(t)$, then the invariant

$$I = L_2' + (L_2'' - L_0)^\circ - c_k^* \partial (L_2' + L_1') / \partial x_k^{\cdot} - \int_0^t h^*(\tau)\, d\tau \tag{3.4}$$

exists along the motion being examined $z = r(t)$, $z^{\cdot} = v(t)$ of subsystem S_2^*. Systems of form S^* include the one considered in [1]. They generalize the latter in the following respects. For them the force function is examined in the general form and the conditions $L_1' \equiv 0$ and $L_1'' \equiv 0$ are not needed. In addition, the particular motion $\mathbf{r}(t)$, $\mathbf{v}(t)$ is taken in the general form for S_2^* and the part of expression (3.4), linear with respect to the momenta, is not necessarily a projection [1] of the moment of momentum of subsystem S_1^*.

4. Examples. Using the results obtained, let us determine the form and the existence conditions for invariant (3.4) for two gyrostatic systems in the case S^*. As a first example we consider a gyrostat S_1 moving in the gravitational field of a spheroid with a fixed center of mass o_2. We assume [3] that the spheroid, a rigid body, rotates around a fixed symmetry axis $o_2\gamma$. The unit vector $\boldsymbol{\gamma}$ lies in the plane $o_2\xi\zeta$ of a fixed trihedron $o_2\xi\eta\zeta$ and makes a constant angle i with the unit vector $\boldsymbol{\zeta}$ ($\cos i = \boldsymbol{\zeta}\cdot\boldsymbol{\gamma}$ is the scalar product of $\boldsymbol{\zeta}$ and $\boldsymbol{\gamma}$). The gyrostat S_1 is formed [7, 8] by a nondeformable shell S_1^0 containing a 2ν-dimensional holonomic stationary system S_2^0 with a constant mass distribution in S_1^0. The position of the principal central trihedron $o_1e_1e_2e_3$ of gyrostat S_1 relative to $o_2\xi\eta\zeta$ is determined by the radius-vector $z = (z_1, z_2, z_3)^*$ of the center of mass of S_1 with the projections z_j onto $o_2\xi\eta\zeta$ and with the Euler angles $\psi = \varphi_1$, $\varphi = \varphi_2$, $\theta = \varphi_3$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^*$ $(j = 1, 2, 3)$. The projections of the vectors $\sigma = z\,|\,z\,|^{-1}$ and $\boldsymbol{\zeta}$ and of the angular velocity $\boldsymbol{\omega}$ of shell S_1^0 onto $o_1\, e_1\, e_2\, e_3$ are denoted by σ_k, ζ_k, ω_k $(k = 1, 2, 3)$. The position of system S_2^0 in S_1^0 is given by the vector $\mathbf{q} = (q_\alpha)^*$, $\alpha = 1, 2, \ldots, \nu$.

Let us assume that the internal forces in S_1 are determined by a potential N_1 of the form

$$N_1 = m_k(\varphi_2, \varphi_3, \mathbf{q})\,\omega_k + m_\alpha'(\varphi_2, \varphi_3, \mathbf{q})\,q_\alpha^{\cdot} + N_0(\boldsymbol{\varphi}, \mathbf{q}, z) \tag{4.1}$$

while the forces external to S_1 have the Newtonian gravitational potential $-U_1(\varphi, z)$. We accept the usual inequalities $l\,|z|^{-1} \ll 1$, $lR_0^{-1} \ll 1$, $m_0 \ll M_0$, where m_0 and l are the mass and maximum dimension of S_1, and M_0 and R_0 are the mass and the polar radius of the spheroid. Then with great accuracy we can assume that the rotational motion of S_1 has no influence on its translational motion and that the motion of S_1 has no influence on the rotation of the spheroid. Using this, we separate S_1 into a subsystem S_1' with vector $x = (\varphi_j, q_\alpha)^*$ and a subsystem S_1'' for which $z = r(t)$, $z^\cdot = v(t)$ is the known motion of the center of mass of S_1. The Lagrangian for S_1 has the form

$$
\begin{aligned}
&L = {}^1/_2 G_k\omega_k^2 + k_j\omega_j + T_2 + {}^1/_2 m_0\,|z^\cdot|^2 + N_1 + U_1 \qquad (4.2)\\
&L_2' = {}^1/_2 G_k\omega_k^2 + k_j\omega_j + T_2, \quad L_1' = m_k\omega_k + m_\alpha' q_\alpha^\cdot\\
&L_2'' = {}^1/_2 m_0\,|z^\cdot|^2, \quad L_1'' = 0, \quad L^* = 0, \quad L_0 = N_0 + U_1\\
&k_j = g_{j\alpha}(\mathbf{q})\, q_\alpha^\cdot\\
&T_2 = {}^1/_2 a_{\alpha\beta}(q)\, q_\alpha^\cdot q_\beta^\cdot, \quad \|a_{\alpha\beta}\| > 0 \quad (k, j = 1, 2, 3, \ \alpha, \beta = 1, 2, \ldots \nu)
\end{aligned}
$$

($0 < G_k$ are the principal moments of inertia of S_1). For the separation being examined we obtain the identities (3.1) with due regard to (4.1) and (4.2). Since $\partial L_2' / \partial\psi = \partial L_1' / \partial\psi = 0$, we have that $\xi_1 = c_1\psi$ is the ignorable coordinate of functions L_2' and L_1'. Consequently, S_1 is a subcase of system S^*. Using equalities (3.4), (4.2) and conditions (3.3), we reach the following conclusion.

If a function $h = h_*(t)$ and a constant $c_1 = c_1^* \neq 0$ exist for which the inequalities

$$
\begin{aligned}
&\partial W / \partial t - c_1^* \partial W / \partial\psi + h_*^\cdot(t) = 0 \quad (W = v_j(\partial(N_0 + U_1) / \partial z_j)^\circ) \qquad (4.3)\\
&W - c_1^* \partial V / \partial\psi + h_*(t) = 0 \ (V = L_0^\circ = (N_0 + U_1)^\circ)
\end{aligned}
$$

are satisfied along the motion $r(t)$, $v(t)$, then the rotational motion of gyrostat S_1 has the invariant

$$
\begin{aligned}
&{}^1/_2 G_k\omega_k^2 + k_j\omega_j + T_2 + {}^1/_2 m_0\,|v|^2 + c_1^*\zeta_j(M_j + m_j) - L_0^\circ - \int_0^t h_*(\tau)\, d\tau \qquad (4.4)\\
&\mathbf{M} = (G_1\omega_1 + k_1, \ G_2\omega_2 + k_2, \ G_3\omega_3 + k_3)^* \qquad (k, j = 1, 2, 3)
\end{aligned}
$$

Here we have used notation (4.1) – (4.3) and the equalities

$$\partial L_2' / \partial\psi^\cdot = \mathbf{M}\cdot\boldsymbol{\zeta}, \quad c_m^* = 0 \quad (m = 2, 3, \ldots, \nu + 3)$$

For S_1 we consider the case when S_2 is the Joukowski-Volterra gyrostat [7, 8, 10]. S_2 is a shell supporting three rotors whose axes have been fastened along o_1e_1, o_1e_2, o_1e_3. Let the shell act on the rotors only by pressure forces on their rotation axes. For S_2 the functions (4.2) and (4.1) have the form

$$
\begin{aligned}
&L = {}^1/_2(A_k\omega_k^2 + g_k^{-1}p_k^2 + m_0\,|\mathbf{v}|^2) + U_1, \ N_1 \equiv 0 \qquad (4.5)\\
&0 < g_k = \mathrm{const}, \quad A_k = G_k - g_k > 0, \quad A = \mathrm{diag}(A_1, A_2, A_3)\\
&\mathbf{p} = g\boldsymbol{\omega} + \mathbf{k}, \ \mathbf{k} = g\dot{q}^\cdot\\
&T_2 = {}^1/_2 g_j^{-1}k_j^2, \quad g = \mathrm{diag}(g_1, g_2, g_3), \quad \mathbf{q} = (q_1, q_2, q_3)^*, \quad \mathbf{p}_k(t) = \mathbf{p}_k(t_0)
\end{aligned}
$$

Using an approximate expression for the spheroid's gravitational potential [3], for $M_0^{-1}R_0^{-2}(C_0 - A_0) \ll 1$ we obtain the asymptotics U_* of force function U_1

$$
\begin{aligned}
&U_* = \mu\,|z|^{-1}\{m_0\,[1 + {}^1/_2(C_0 - A_0)M_0^{-1} \times |z|^{-2}(1 - 3s^2)] + \qquad (4.6)\\
&\quad |z|^{-2}(G_0 - {}^3/_2 P)\}\\
&\mu = fM_0, \quad C_0 > A_0, \quad s = \gamma_k\sigma_k = |z|^{-1}\gamma_k z_k, \quad 2G_0 = G_1 + G_2 + G_3, \quad P = G_k\sigma_k^2
\end{aligned}
$$

Here f is Gauss' constant, C_0 and A_0 are the spheroid's moments of inertia relative to the rotation axis $o_2\gamma$ and to the equatorial axis.

Let us consider the following variant of the restricted [9, 10] problem of the translation-rotational motion of gyrostat S_2. We assume that the center of mass of S_2 moves with Keplerian angular velocity $\omega_0 = \mu^{1/2} r_0^{-3/2}$ on a circle of constant radius $|\mathbf{z}| = r_0$ in the plane $o_2\xi\eta$; the vector γ lies in the plane $o_2\xi\zeta$ and makes a constant angle i with the unit vector $\boldsymbol{\zeta}$. This motion is the approximate solution

$$z_1 = r_1(t) = r_0 \cos\tau, \quad z_2 = r_2(t) = r_0 \sin\tau, \quad z_3 = r_3(t) = 0 \qquad (4.7)$$
$$\tau = \omega_0 (t - t_0) + u_0, \quad t_0 = \text{const}, \ u_0 = \text{const}$$

of the equations of motion of the center of mass of gyrostat S_2

$$m_0 \ddot{z}_j = \partial U_1 / \partial z_j \ (j = 1, 2, 3)$$

which in the restricted formulation can be considered as the exact solution. The latter, together with U_1 replaced by function (4.6), serves as the initial assumptions of the variant being examined of the restricted problem.

We set
$$c_1^* = -\omega_0, \ h_* = {}^3/_2\omega_0^3 (C_\theta - A_0) m_0 M_0^{-1} \sin^2 i \sin 2\tau$$

into conditions (4.3). Substituting expressions (4.5) — (4.7) into them, we see that equalities (4.3) are satisfied. Therefore, the particular Jacobi invariant [1]

$$I = {}^1/_2 A_k \omega_k^2 - \omega_0 \zeta \cdot (A\omega + p) + {}^3/_2 \omega_0^2 G_k \sigma_k^2 \qquad (4.8)$$

exists in the case being examined for the motion of S_2 We obtain expression (4.8) from (4.4) with due regard to equalities (4.5) — (4.7). Thus, in this variant of the motion of S_2 the existence conditions (4.3) for invariant (4.8) are satisfied. We note that expression (4.8), obtained under the asymptotics (4.6) of the noncentral gravitational field, coincides with the generalized energy integral [9, 10] in the case of a central field.

REFERENCES

1. Jacobi, C. G., Vorlesungen über Dynamik. Berlin, Reimer, 1884.
2. Gantmakher, F. R., Lectures on Analytical Mechanics. Moscow, "Nauka", 1966.
3. Demin, V. G., Motion of an Artificial Satellite in a Noncentral Gravitational Field. Moscow, "Nauka", 1968.
4. Leech, J. W., Classical Mechanics. London, Methuen and Co., Ltd., 1958.
5. Goursat, É., Cours d'Analyse Mathématique, T. 11, Paris, Gauthier-Villars, 1927.
6. Eisenhart, L. P., Continuous Groups of Transformations. London, Oxford Univ. Press, 1933.
7. Volterra, V., Sur la théorie des variations des latitudes. Acta Math., Vol. 22, pp. 201-358, 1899.
8. Rumiantsev, V. V., On the stability of motion of certain types of gyrostats. PMM Vol. 25, № 4, 1961.
9. Beletskii, V. V., Some questions on the translation-rotational motion of a rigid body in a Newtonian force field. In: Artificial Earth Satellites, № 16, Moscow, Izd. Akad. Nauk SSSR, 1963.
10. Rumiantsev, V. V., On the Stability of Steady-State Motions of Satellites. Moscow, Vychisl. Tsentr, Akad. Nauk SSSR, 1967.

Translated by N. H. C.